FIN DIM LIE + ASS ALGEBRAS

1 INTRODUCTION

Dfn 1.1: A **Lie Algebra** is a K-vector space and a bilinear operation $[\cdot,\cdot]: L \times L \to L$ satisfying

(1) $[x,x] = 0 \quad \forall x \in L$

(2) **Jacobi identity:** $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$

Dfn 1.2 (a) A **lie subalgebra** J of L is a K-subspace of L s.t $\forall\, x,y \in J,\ [x,y] \subset J$.

(b) a (lie) **ideal J of L** is a K-subspace s.t $[x,y] \in J \ \forall\, x \in J,\ y \in L$.

note: dfn is symmetric actually in x and y.

Dfn 1.3 (a) L is **semisimple** if $R(L) = 0$ and in general, $L/R(L)$ is semisimple.

(b) L is **simple** $\Leftrightarrow$ the only ideals are 0 and L.

also $[L,L] \neq 0$, to avoid 1-dimensional case.

2 LIE ALGEBRAS

Dfn (associative K-algebra) ring with 1 and $\phi: K \to R$ a ring homomorphism $1 \mapsto 1$ and want that $\phi(K) \subseteq Z(R)$ (centre of R)

Then R is a Lie Algebra via $[r,s] = rs - sr$.

$\mathfrak{gl}_n \cong M_n(K)$ = Lie. alg. assoc. to $GL_n(\mathbb{R})$

(1) matrices of trace $= 0 =: \mathfrak{sl}_n$ is associated with SL_n. $\quad SL_n = \{n \times n \text{ matrices w/ determinant } 1\}$ $\quad \mathfrak{gl}_n \cong M_n(K)$

e.g. $\mathfrak{sl}_2$ standard notation: $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

note that $[e,f] = h$, $[h,e] = 2e$, $[h,f] = -2f$

(2) $\mathfrak{so}_n$ = Skew symmetric $n \times n$ matrices, associated with Special orthogonal group SO_n.

e.g. $n = 3$, $\mathfrak{so}_3$: $A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ $A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Then $[A_1,A_2] = A_3$, $[A_2,A_3] = A_1$, $[A_3,A_1] = A_2$.

(3) $\mathfrak{sp}_{2n}$ = contains matrices associated with symplectic group Sp_{2n} of matrices that preserve a non-degenerate, skew-symmetric product on K^{2n}.

e.g. let's say the skew-symmetric form is represented by $J = \left(\begin{array}{c|c} 0 & \iddots^{\,1} \\ \hline \iddots_{-1} & 0 \end{array}\right)_{2n \times 2n}$

Then $\mathfrak{sp}_{2n}$ consists of matrices X s.t. $XJ + JX^t = 0$

(alternative formulation: take $J = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right)$)

$\to$ these matrices are of form $\left(\begin{array}{c|c} A & B \\ \hline C & -A^T \end{array}\right)$, B and C symmetric.

dimension of $\mathfrak{sp}_{2n}$ is $2n^2 + n$

For alternative J you get $\left(\begin{array}{c|c} A & B \\ \hline C & A^T \end{array}\right)$, B and C skew-symmetric.

(4) $\mathfrak{b}_n$ Borel subalgebra of $\mathfrak{gl}_n$ of upper triangular matrices associated with the Borel subgroup of GL_n consisting of invertible upper triangular matrices.

(5) $\mathfrak{n}_n$ consists of strictly upper triangular matrices associated with the group of upper triangular matrices with 1's on the diagonal.
($\mathfrak{n}$ denotes nilpotent).

$End_K(R)$:= K-linear maps $R \to R$. Lie Algebra w/ $[f,g] = f \circ g - g \circ f$

Dfn 2.1 a linear map $D: R \to R$ is a **derivation** if $D(rs) = D(r)s + rD(s)$ (ring product)

$Der(R) = \{\text{derivations } R \to R\}$ forms a lie subalgebra of $End_K(R)$

when $R = L$, $\cdot$ is $[\cdot,\cdot]$ so derivation: $D([r,s]) = [D(r),s] + [r,D(s)]$

Dfn 2.2 An **inner derivation** of R is of the form $R \to R$, $s \mapsto [r,s]$ for some $r \in R$.

$Innder(R)$ = {inner derivations} forms a lie ideal in $Der(R)$.

R commutative: $[x,y] = xy - yx = 0 \Rightarrow Inder(R) = 0$.

Dfn 2.3 (a) A **lie algebra homomorphism** $\rho: L_1 \to L_2$ is a K-linear map satisfying

$$\rho([x,y]) = [\rho(x), \rho(y)]$$

(b) **lin rep. of L** is a L.A. homo: $\rho: L \to End(V)$ for some V.

$U \leq V$ and $\rho(L)(U) \leq U$, then $\rho|_U: L \to End(U)$ is called a subrepresentation.

(c) **irreducible repn:** only such U are 0 and V.

adjoint repn: $ad_L: L \to End(L)$; $x \mapsto ad(x): L \to L$; $y \mapsto [x,y]$ $\quad (x \mapsto [x,-])$.

Dfn 2.4: The **centre of L** $= \{x : [x,y] = 0 \ \forall y \in L\} = \ker(ad_L)$

Rem: L simple Lie Algebra $\Leftrightarrow$ $ad(L)$ is irreducible

Dfn 2.8: An **Abelian Lie algebra** L if $[x,y] = 0 \ \forall\, x,y \in L$.

Dfn 2.9: The **derived series of L** $L^{(0)} = L$, $L^{(1)} = [L,L] = span\{[x,y] : x,y \in L\}$ (derived subalgebra)

$L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$ $\quad i \geq 2$.

Dfn 2.10: L is **Soluble** if $L^{(r)} = 0$. Least r := "derived length".

Rem: $L^{(i)}$ = ideal of L.

J ideal, then L/J Lie Alg. with bracket $[x+J, y+J] = [x,y] + J$

Lemma 2.11 (1) Subalgebras and quotient of soluble lie algebras are soluble

(2) If J is an ideal of L, then L is Soluble $\Leftrightarrow$ J and L/J are soluble

Ex: any 2-dim. Lie Alg is soluble $\qquad$ Ex: $\mathfrak{so}_3$ is not soluble!

Lemma 2.12: The sum of two soluble ideals is a soluble ideal.

Dfn 2.13: **Radical $R(L)$** of F.D.L.A. L is the max. soluble ideal in L $= \Sigma$ all soluble ideals.

Semisimple $\Leftrightarrow$ $R(L) = 0$.

Thm 2.14 (Levi). If $char K = 0$ and L is finite dimensional, then there exists a Lie subalgebra L_1 such that $L_1 \cap R(L) = 0$, and $L = L_1 + R(L)$

Hence $L_1 \cong L/R(L)$ is semisimple.

Dfn 2.15: This is the **Levi decomposition**, and L_1 is the **Levi factor** (/ subalgebra).

Dfn 2.16: The **lower central series** of L is $L_{(1)} = L$, $L_{(i+1)} = [L_{(i)}, L]$

Note: (1) $L_{(i)}$ are ideals of L
(2) counting starts at 1.

L is **nilpotent** if $L_{(c)} = 0$, $c =:$ **nilpotency class**

Proposition (page 12 of Humphrey's): Let L be a Lie Algebra.
a) if L is nilpotent, then so are all subalgebras and homomorphic images of L
b) if $L/Z(L)$ is nilpotent, then so is L.
c) if L is nilpotent and nonzero, then $Z(L) \neq 0$.

Thm 2.18: (Lie) For algebraically closed k, $char K = 0$. Suppose $L \leq End(V)$ with $\dim V < \infty$. Suppose **L is Soluble. Then $\exists\, v \in V$, $v \neq 0$, such that $x(v) = \lambda v$ for all $x \in L$.** (v is a common eigenvector $\forall\, x \in L$)

Thm 2.19: (Engel) Suppose $L \leq End(V)$ is a Lie subalgebra, $\dim V < \infty$, and every element of L is a nilpotent endomorphism (i.e. $\forall\, x \in L$, $\exists\, a \in \mathbb{N}$ s.t $x^a = 0$). **Then $\exists\, v \neq 0$, $v \in V$ such that $x(v) = 0 \ \forall x \in L$.**

Corollary: $ad(L)$ is then a nilpotent Lie algebra, $\Rightarrow L$ is nilpotent

$((ad(L))_{(c)} = ad(L_{(c)})$ by induction + ad is a L.A. homo.

L soluble $\Rightarrow$ $\exists$ basis of V wrt which L is represented by upper triangular matrices: $L \subseteq \mathfrak{b}_n$

$\Rightarrow L^{(1)} \subseteq \mathfrak{n}_n$ = strictly upper triangular matrices.

Q7: $L^{(m)}$ lies in $L_{(2^m)}$ $\forall$ positive m for a Lie Algebra L.

Nilpotent Lie Algebras are soluble $\qquad$ Semisimple $\Rightarrow L \cong ad(L)$

3 INVARIANT FORMS + CARTAN - KILLING CRITERION.

Dfn 3.1: A symmetric bilinear form $\langle\cdot,\cdot\rangle : L \times L \to K$ is **invariant** if
$$\langle [x,y], z\rangle = \langle x, [y,z]\rangle.$$

Dfn 3.2: (a) if $\rho : L \to \mathrm{End}(V)$ with $\dim V < \infty$ is a representation, then
$$\langle x,y\rangle_\rho = \mathrm{Tr}\,(\rho(x)\rho(y))$$
composition of endomorphisms.

is the **trace form of ρ.**

(b) The trace form of the adjoint representation (when $\dim L < \infty$) is the **killing form.**

Lemma 3.3: (i) trace forms are invariant symmetric bilinear forms.
(ii) If J is an ideal, then $J^\perp = \{x : \langle x,y\rangle = 0 \ \forall y \in J\}$ and for an invariant form $\langle\cdot,\cdot\rangle$, then $J^\perp$ is an ideal. In particular, $L^\perp$ is an ideal of L.

Thm 3.4 (Cartan's criterion for solubility). Let $\mathrm{char}\,K = 0$, and L be a Lie subalgebra of $\mathrm{End}(V)$, $\dim V < \infty$. Let $\langle,\rangle$ be trace form of the embedding $\rho : L \to \mathrm{End}(V)$. Then
L is soluble $\Leftrightarrow$ $\langle x,y\rangle_\rho = 0 \ \forall x \in L, \ y \in L^{(1)}$.

Thm 3.5: (Cartan - Killing criterion for semisimplicity). Let $\mathrm{char}\,K = 0$. Then
L is semisimple $\Leftrightarrow$ The killing form $\langle\cdot,\cdot\rangle_{ad}$ is non-degenerate.

Dfn 3.6: A **derivation** of a lie algebra L is a linear map $L \to L$ such that
$$D([x,y]) = [x, Dy] + [Dx, y].$$

Inner derivations are of the form $y \mapsto [x,y]$.
$$\{\text{Inner derivations}\} = \mathrm{ad}(L)$$

Thm 3.7 If $\mathrm{char}\,K = 0$ and $\dim L < \infty$, and L is semisimple, then $\mathrm{Der}(L) = \mathrm{ad}(L)$.

Dfn 3.8: $x \in \mathrm{End}(V)$ is semisimple $\Leftrightarrow$ it is **diagonalizable.**
$\Leftrightarrow$ minimal polynomial is a product of distinct linear factors.

Rem 1) if x is semisimple, $x(W) \le W$ for a subspace $W \le V$, then $x|_W : W \to W$ is semisimple
2) if x,y semisimple and $xy = yx$, then x,y are simultaneously diagonalizable, and $x \pm y$ is also semisimple.

Lemma 3.9 (Jordan decomposition)
For $x \in \mathrm{End}(V)$,
(1) $\exists$ unique $x_s, x_n \in \mathrm{End}(V)$ with x_s semisimple, x_n nilpotent, and x_n, x_s commute, and $x = x_s + x_n$.
(2) $\exists$ polynomials $p(t), q(t)$ with zero constant term such that $x_s = p(x)$, and $x_n = q(x)$. So, x_s, x_n commute with all endomorphisms that commute with x.
(3) If $U \le W \le V$ and $x(W) \subseteq U$, then $x_s(W) \subseteq U$ and $x_n(W) \subseteq U$.

Lemma 3.11: Let $x = x_s + x_n \in L \subseteq \mathrm{End}(V)$. Then $\mathrm{ad}(x) = \mathrm{ad}(x_s) + \mathrm{ad}(x_n)$
$\mathrm{ad}(x_s)$ = semisimple part, $\mathrm{ad}(x_n)$ = nilpotent part

Proposition 3.13: Let L be a finite dimensional Lie algebra, $\mathrm{Char}\,K = 0$.
(1) if L is semisimple, then L is a direct sum of non abelian, simple ideals.
(2) if $0 \neq J$ is an ideal of $L = \bigoplus L_i$, then the ideal is a direct sum of some of the L_i
(3) if L is a direct sum of non abelian simple ideals, then L is semisimple.

4 CARTAN SUBALGEBRAS AND WEIGHT DECOMPOSITION

Dfn 4.1:
$$L_{\lambda,y} = \{x \in L : (\mathrm{ad}(y) - \lambda \overset{\text{identity map}}{i})^r(x) = 0\}$$
is the **generalized λ-eigenspace for $\mathrm{ad}(y)$** $(y \neq 0)$.

Lemma 4.2:
(i) $[L_{\lambda,y}, L_{\mu,y}] \subseteq L_{\lambda+\mu,y}$ (ii) $L_{0,y}$ is a Lie subalgebra

Dfn 4.3: A **Cartan subalgebra (CSA)** H of L is nilpotent and self idealising:
$$\underbrace{\{x : [x,H] \subseteq H\}}_{\text{idealiser}} = H$$
$L(c) = 0$ for some $c \in K$

Theorem 4.4 (Cartan) [Existence of CSAs]
H is a Cartan subalgebra $\Leftrightarrow$ H is a minimal subalgebra of the form $L_{0,y}$.
All CSAs have the same dimension

Thm 4.6 (not proved here)
Any two CSAs are conjugate under the group of automorphisms of L, which are generated by $e^{\mathrm{ad}(y)} = 1 + \mathrm{ad}(y) + \frac{(\mathrm{ad}(y))^2}{2!} + \dots$ with $\mathrm{ad}(y)$ nilpotent (i.e. finite sum).

Thm 4.7 (not proved here)
The set of regular elements (elements $y \in L$ s.t $L_{0,y}$ is a CSA) is connected.
Ie. Zariski dense, open subset of L

Theorem 4.8: Let H be a CSA of a semisimple L. Then
(a) it is a maximal abelian subalgebra
(b) every element of H is semisimple.
(c) The restriction of the killing form $\langle\cdot,\cdot\rangle_{ad}$ of L to H is also nondegenerate.

Lemma 4.9: (converse of 4.8). Let H be a maximal abelian subalgebra $\subseteq L$, all of whose elements are semisimple. Then H is a Cartan subalgebra.

Corollary 4.10 (of 4.8): Regular elements of semisimple L are semisimple

$$L_\alpha := \{x \in L : \mathrm{ad}(h)(x) = \alpha(h)(x) \ \forall h \in H\}$$
$\alpha : H \to \mathbb{C}$ is a linear form.

Dfn 4.11: The weight space or Cartan decomposition of semisimple L wrt. CSA H.
$$L = L_0 \oplus \Big(\bigoplus_{\alpha \neq 0} L_\alpha\Big) \quad \text{with} \quad L_0 = H$$

The nonzero elements of L_α have <u>weight</u> α
The $L_\alpha \neq 0$ are the <u>weight spaces</u>
The nonzero weights are called the <u>roots</u> of L (wrt H).

Notation: Φ = set of roots.
$m_\alpha = \dim L_\alpha$
$\langle\cdot,\cdot\rangle$ = killing form

Lemma 4.12
(a) $x,y \in H \Rightarrow \langle x,y\rangle = \sum_{\alpha\in\Phi} m_\alpha\, \alpha(x)\,\alpha(y)$
(b) $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$
(c) $\langle\cdot,\cdot\rangle$ restricted to H is non-degenerate
(d) If α,β weights and if $\alpha + \beta \neq 0$, then $\langle L_\alpha, L_\beta\rangle = 0$.
(e) $\alpha \in \Phi \Rightarrow -\alpha \in \Phi$
(f) α weight $\Rightarrow L_\alpha \cap L_{-\alpha}^\perp = 0$
(g) If $0 \neq h \in H$, then $\alpha(h) \neq 0$ for some $\alpha \in \Phi$.
So Φ spans H^*, dual space of H.

Dfn 4.13 The **α-string through β** is the largest arithmetic progression
$\left(\begin{array}{l}\alpha \text{ is a root,} \\ \beta \text{ is a weight}\end{array}\right)$
$$\beta - q\alpha, \dots, \beta, \dots, \beta + p\alpha$$

Lemma 4.14: For $\alpha \in \Phi$, β weight, p,q as above. Then

(a)
$$\beta(x) = -\left(\frac{\sum_{r=-q}^{p} r\, m_{\beta+r\alpha}}{\sum_{r=-q}^{p} m_{\beta+r\alpha}}\right)\alpha(x) \quad \text{for } x \in [L_\alpha, L_{-\alpha}].$$

(b) if $0 \neq x \in [L_\alpha, L_{-\alpha}]$, then $\alpha(x) \neq 0$

(c) $[L_\alpha, L_{-\alpha}] \neq 0$.

Write $\begin{cases} h \in H, \ h^* \text{ by } h^*(x) = \langle h, x\rangle_{ad} \\ h_\alpha \text{ for the preimage of } \alpha \in H^*. \end{cases}$

Dfn 4.17: $(\alpha,\beta) := \langle h_\alpha, h_\beta\rangle_{ad}$ for $\alpha,\beta \in H^*$.

Where $\begin{cases} \langle h_\alpha, x\rangle = \alpha(x) \\ \langle h_\beta, x\rangle = \beta(x) \end{cases} \ \forall x \in H$

Lemma 4.19: For $\alpha,\beta \in \Phi$,

(a) $\dfrac{2(\beta,\alpha)}{(\alpha,\alpha)} = \dfrac{2\langle h_\beta, h_\alpha\rangle}{\langle h_\alpha, h_\alpha\rangle} \in \mathbb{Z}$

(b) $4 \sum_{\beta\in\Phi} \dfrac{\langle h_\beta, h_\alpha\rangle^2}{\langle h_\alpha, h_\alpha\rangle^2} = \dfrac{4}{\langle h_\alpha, h_\alpha\rangle} \in \mathbb{Z}$

(c) $\langle h_\alpha, h_\beta\rangle \in \mathbb{Q} \quad \forall \alpha,\beta \in \Phi.$

(d) $\forall \alpha,\beta \in \Phi$, $\beta - \dfrac{2\langle h_\beta, h_\alpha\rangle}{\langle h_\alpha, h_\alpha\rangle}\alpha \in \Phi$.

5 ROOT SYSTEMS

Dfn 5.1: a subset Φ of a real Euclidean vector space E is a **finite root system** if
(i) Φ is finite, spanning E and not containing 0.
(ii) for each $\alpha \in \Phi$, there's a **reflection** s_α (preserving the inner product) with $s_\alpha(\alpha) = -\alpha$, the set of fixed points is a hyperplane of E, and s_α preserves Φ.
(iii) for each $\alpha, \beta \in \Phi$, $s_\alpha(\beta) - \beta$ is an integral multiple of α.
(iv) for $\alpha, \beta \in \Phi$, $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$
(v) $s_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \quad \forall \beta \in \Phi$

Dfn 5.2: The **rank** of a root system = dim E.

Dfn 5.3: A root system is **reduced** if for each $\alpha \in \Phi$, the only roots proportional to α are $\pm\alpha$.

Dfn 5.4: The **Weyl group** $W(\Phi)$ of a root system is a subgroup of the orthogonal group generated by the reflections s_α, $\alpha \in \Phi$.

Dfn 5.5: for a finite root system, write $n(\beta,\alpha)$ for $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$. Let $|\alpha| = (\alpha,\alpha)^{1/2}$.
Then $(\alpha,\beta) = |\alpha||\beta|\cos\phi$, where ϕ is an angle between α, β.

Then $n(\beta,\alpha) = \frac{2|\beta|}{|\alpha|}\cos\phi$.

$n(\alpha,\beta)$	$n(\beta,\alpha)$	ϕ	Notes
0	0	$\frac{\pi}{2}$	
1	1	$\frac{\pi}{3}$	$\lvert\beta\rvert = \lvert\alpha\rvert$
-1	-1	$\frac{2\pi}{3}$	$\lvert\beta\rvert = \lvert\alpha\rvert$
1	2	$\frac{\pi}{4}$	$\lvert\beta\rvert = \sqrt{2}\lvert\alpha\rvert$
-1	-2	$\frac{3\pi}{4}$	$\lvert\beta\rvert = \sqrt{2}\lvert\alpha\rvert$
1	3	$\frac{\pi}{6}$	$\lvert\beta\rvert = \sqrt{3}\lvert\alpha\rvert$
-1	-3	$\frac{5\pi}{6}$	$\lvert\beta\rvert = \sqrt{3}\lvert\alpha\rvert$

possible reduced root systems.

Dfn 5.7: An **isomorphism of a root system** $(E,\Phi) \longrightarrow (E',\Phi')$ is a linear bijection such that $\phi(\Phi) = \Phi'$

Dfn 5.8: (a) The direct sum of two root systems (E,Φ) and (E',Φ') is $(E \oplus E', \Phi \cup \Phi')$
(b) A root system that is not isomorphic to a direct sum of root systems is called **irreducible**.

Dfn 5.9: if $\alpha \in \Phi$, define the **co-root** $\alpha^\vee = \frac{2}{(\alpha,\alpha)}\alpha$. Thus $(\alpha,\alpha^\vee) = 2$.

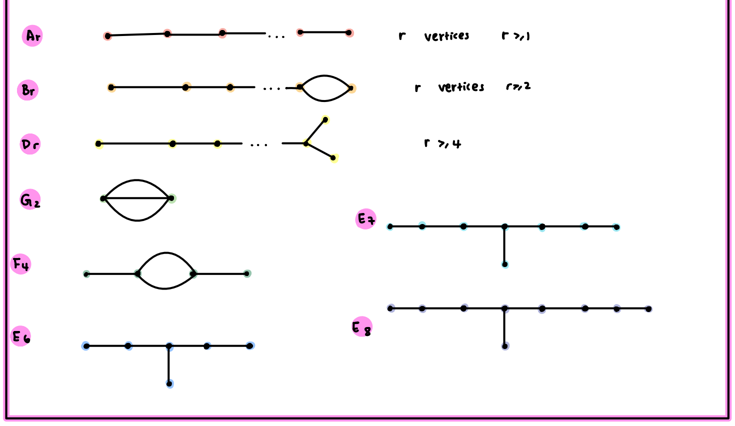
Dfn 5.10: A root system is **simply laced** if all the roots are of the same length.

Dfn 5.11: A subset Δ of a root system (E,Φ) is a **base of Φ** if
1) Δ is a vector space basis for E
2) each $\gamma \in \Phi$ can be written as a linear combination
$$\gamma = \sum_{\alpha\in\Delta} k_\alpha \alpha$$
with coefficients k_α integers and either **all ≥ 0 or all ≤ 0.**

Dfn 5.12: The **Cartan matrix** of a root system wrt. Δ is the matrix $(n(\alpha,\beta))_{\alpha,\beta\in\Delta}$

Dfn 5.13: A **coexeter graph** is a finite graph, each pair of vertices connected by 0, 1, 2 or 3 edges.
Given a root system Φ with base Δ, the coexeter graph of (E,Φ) wrt Δ has:
- vertices: elements of Δ = simple roots
- vertex α is joined to β for 0, 1, 2, 3 according to $n(\alpha,\beta)n(\beta,\alpha) = 0,1,2,3$

Theorem 5.14: Every connected, nonempty coexeter graph associated with a root system arising from a semisimple complex Lie algebra is isomorphic to



arrow pointing towards shorter root.
The graphs with arrows are called **Dynkin Diagrams.**

$\Phi^+(\gamma) = \{\alpha \in \Phi : (\gamma,\alpha) > 0\}$.

Let $\gamma \in E$. Then γ *regular* if $\gamma \in E \setminus \bigcup_\alpha P_\alpha$

$\Rightarrow \Phi = \Phi^+(\gamma) \cup (-\Phi^+(\gamma))$

$\alpha \in \Phi^+(\gamma)$ is *indecomposable* if not expressable as $\alpha = \alpha_1 + \alpha_2$, $\alpha_1, \alpha_2 \in \Phi^+(\gamma)$, $\alpha_1 \neq \alpha_2$

$\Delta(\gamma) = \{$ indecomposable elts of $\Phi^+(\gamma)\}$

Lemma 5.16: $\Delta(\gamma)$ is a base, and every base is of this form.

Lemma 5.17: Δ reduced Φ:
(a) $\alpha, \beta \in \Delta$ distinct, then $(\alpha,\beta) \leq 0 \Rightarrow$ nondiag in Cartan ≤ 0
(b) $\alpha \in \Phi^+$, $\alpha \notin \Delta$, then $\exists\, \beta \in \Delta$ s.t $\alpha - \beta \in \Phi^+$
(c) each $\alpha \in \Phi^+$ is of form $\beta_1 + \dots + \beta_n$, each $\beta_1 + \dots + \beta_i \in \Phi^+$
(d) α simple, then α permutes $\Phi^+ \setminus \{\alpha\}$

Lemma 5.18: Δ simple roots
(a) if $\sigma \in GL(E)$ orthog + $\sigma(\Phi) = \Phi$, then $\sigma s_\alpha \sigma^{-1} = s_{\sigma(\alpha)}$
(b) Let $\alpha_1, \dots, \alpha_t \in \Delta$ (not necessarily distinct). Write s_i for s_{α_i}. If $s_t \cdots s_2(\alpha_1)$ negative, or equiv. $s_t \cdots s_1(\alpha_1)$ positive, then for some i,
$$s_t \cdots s_1 = s_t \cdots \hat{s}_i \cdots \hat{s}_1$$
(c) $\sigma = s_t \cdots s_1$ expression of an elt in W with t minimal, then $\sigma(\alpha_1)$ is negative.

Lemma 5.19: $W = W(\Phi)$, Φ reduced.
(a) γ regular, then $\exists\, \sigma \in W$ with $(\sigma(\gamma),\alpha) > 0 \quad \forall \alpha \in \Delta$
↳ W permutes bases transitively
(b) $\alpha \in \Phi$, then $\sigma(\alpha) \in \Delta$ for some $\sigma \in W$
(c) $W = \langle s_\alpha$ for $\alpha \in \Delta \rangle$
(d) If $\sigma(\Delta) = \Delta$, then $\sigma = 1$.

Theorem 5.20: (not proved here): $W(\Phi) = \langle \underbrace{s_\alpha}_{\text{simple reflections}} : s_\alpha^2 = 1,\ (s_\alpha s_\beta)^{m(\alpha,\beta)} = 1 \rangle$ (order of $s_\alpha s_\beta$)

6 REPRESENTATION THEORY OF SEMISIMPLE COMPLEX LIE ALGEBRAS

Theorem 6.1 (Weyl): Let L be a semisimple, finite dimensional Lie algebra, char k = 0. Then all finite dimensional representations are a direct sum of irreducible ones.

Definition 6.2 A representation is **completely reducible** if it is such a direct sum

Lemma 6.3: The following are equivalent:
(1) all finite dimensional representations are completely ~~i~~rreducible.
(2) whenever $\rho : L \to \mathrm{End}(V)$ with $W \subseteq V$ and $\dim(V/W) = 1$, and $\rho(L)(V) \subseteq W$ (W is invariant), then there is a W' with $V = W \oplus W'$ and $\rho(L)(W') \subseteq W'$.
(3) The same as (ii) but with the restriction of ρ to W, $\rho_W : L \to \mathrm{End}(W)$, is irreducible.

Casimir element of representation ρ $\quad c = \sum \rho(x_i)\rho(y_i) \in \mathrm{End}(V)$.

Lemma 6.4 $[c, \rho(z)] = 0 \quad \forall z \in L$. (commute in End(V)).

Universal Enveloping Algebra

Dfn 6.5: $\mathcal{U}(L)$ is the associative algebra with generators $X\ (\in L)$ and relations $\underbrace{XY - YX}_{\text{commutator of } X,Y \text{ in enveloping algebra}} = \underbrace{[X,Y]}_{\text{lie bracket in } L}$. for $X,Y \in L$

Theorem 6.6 (Poincaré - Birkhoff - Witt PBW): $\mathcal{U}(L)$ has a basis as a $\mathbb{C}$-vector space $\{x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} : m_i \in \mathbb{Z}_{\geq 0}\}$ where $x_1, \dots, x_n$ is a basis.

$$\left\{\begin{matrix} \rho: L \to \text{End}(V) \\ \text{representations} \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \bar{\rho} : \mathcal{U}(L) \to \text{End}(V) \\ V \text{ is a } \mathcal{U}(L)\text{-module} \end{matrix}\right\}$$

Dfn 6.7: Let $V_\omega = \{v \in V : \rho(h)(v) = \omega(h)v \ \ \forall h \in H\}$ be the weight space of weight ω, where $\omega \in H^*$.

Lemma 6.8
a) $\rho(L_\alpha)V_\omega \subseteq V_{\omega+\alpha}$ if $\omega \in H^*$, $\alpha \in \Phi$
b) The sum of the V_ω is direct and is invariant under $\rho(L)$.
c) (assuming L is semisimple) if $\dim(V) < \infty$ then V = direct sum of weight spaces.

$N = \sum_{\alpha \in \Phi^+} L_\alpha$, and $N^- = \sum_{\alpha \in \Phi^+} L_{-\alpha}$ $\qquad L = N^- \oplus H \oplus N = N^- \oplus B$

$B = H \oplus N$

Dfn 6.9: v is a primitive element of weight ω if it satisfies
(i) $v \neq 0$, has weight ω $(\omega \in H^*)$
(ii) $\rho(N)(v) = 0$

Proposition 6.10 Let v be a primitive element of weight ω, and let $W = \rho(L)(v)$. Then
(i) W is spanned by $\rho(y_1)^{m_1} \cdots \rho(y_k)^{m_k}(v)$, where the β_i are the distinct +ve roots and $m_i \in \mathbb{Z}_{\geq 0}$
(ii) the weights of W are of the form $\omega - \sum_{i=1} p_i \alpha_i$, where $\{\alpha_1, \dots, \alpha_r\}$ is a base with $p_i \in \mathbb{Z}_{\geq 0}$, and they have finite multiplicity (weight spaces are finite dim)
(iii) ω has multiplicity 1, and the weight space in W of weight $\omega = \langle v \rangle$.
(iv) $\rho_W : L \to \text{End}(W)$ is indecomposable. I.e. W cannot be expressed nontrivially in the form of a direct sum $W_1 \oplus W_2$, with W_i invariant.

not necessarily finite dimensional

Theorem 6.11: Let V be a simple $\mathcal{U}(L)$-module (ρ is an irreducible representation) and suppose V contains a primitive element v of weight ω.
a) v is the only primitive element of V up to scalar multiplication.
b) The weights of V have the form $\omega - \sum p_i \alpha_i$ with $p_i \in \mathbb{Z}_{\geq 0}$. They have finite multiplicities, and ω has multiplicity 1, and V is a sum of the weight spaces.
c) For two simple modules V_1 and V_2, with primitive elements v_1, and v_2 of weight ω_1 and ω_2 respectively, then $V_1 \cong V_2$ iff $\omega_1 = \omega_2$.

Dfn 6.12: The weight ω of the primitive element V is known as the highest weight.

Theorem 6.13: For each $\omega \in H^*$ there is a simple $\mathcal{U}(L)$-module of highest weight ω.

7 FINITE DIMENSIONAL ASSOCIATIVE ALGEBRAS

Dfn 7.1: R is a simple (associative) algebra if its only two-sided ideals are 0 and R.

Dfn 7.2: The Jacobson radical $J(R) = \cap\{\text{max}^l \text{ proper right ideals}\}$

Note: I is a maximal right ideal $\Leftrightarrow$ R/I is a simple right R-module.

$\text{Ann}_R(m) : \{r \in R : mr = 0\}$

$J(R) = \bigcap_{\substack{M \text{ simple} \\ \text{right modules}}} \text{Ann}_R(M) = 2 \text{ sided ideal.}$

Lemma 7.2 (Nakayama)
The following are equivalent: for a right ideal I,
(i) $I \subseteq J(R)$
(ii) If M is a f g R-module and $N \subseteq M$ w/ $N + MI = M$, then $N = M$
(iii) $\{1 + x : x \in I\} = G$ is a subgroup of the unit group of R $(R^\times)$

Dfn 7.3: R is semisimple if $J(R) = 0$.

Lemma 7.4: Let R be a semisimple, fin dim (ass) algebra. Then R is the direct sum of finitely many simple (right) R-modules.

Lemma 7.5: let R be semisimple, M any nonzero, fin dim R module, then M is a direct sum of simple modules
↳ a quotient of $R \oplus \cdots \oplus R \to M$ simple then a quotient of R.

Definition 7.6: M is completely reducible (semisimple) is it can be written as a direct sum of simple R-modules.

Definition 7.7: The socle of a fin dim R module M: $\text{soc}(M) := \sum\{\text{min. nonzero submodules of } M\}$

Lemma 7.8: $\text{Soc}(M) = \{m \in M : mJ(R) = 0\}$

Definition 7.9: The socle series of M:

$$0 \subseteq \text{Soc}_0(M) \subsetneq \text{Soc}_1(M) \subsetneq \cdots$$

where $\text{Soc}_i(M)/\text{Soc}_{i-1}(M) = \text{soc}\left(M/\text{soc}_{i-1}(M)\right)$

Remark: 1) The series must terminate at M.
2) $\text{Soc}_i(M) = \{m \in M : mJ^i = 0\}$

Proposition 7.10: Let R be a fin dim (ass) algebra. Then $J(R)$ is nilpotent (i.e $\exists\, m \in \mathbb{Z}_{>0}$ s.t $J^m = 0$).

Lemma 7.11: (Schur's Lemma)
Let S be a simple right R-module. Then $\text{End}_R(S)$ is a division ring.
If S_1, S_2 are non-iso simple R-modules, then $\text{Hom}_R(S_1, S_2) = \{0\}$.

Lemma 7.12: Regarding R as a right R-module (R_R), then $\text{End}(R_R) \cong R$ via multiplication on the left by elements of R.

$$\text{End}(R_R) \leftrightarrow R, \quad \phi \longmapsto \phi(1)$$
$$R \to \text{End}(R), \quad r \longmapsto r\cdot (R \to R) \ ; \ r \cdot x = rx$$
$$\phi(1) \longmapsto \phi(1)\cdot x = \phi(x).$$

Theorem 7.13: (Artin - Wedderburn) Let R be a semisimple fin dim ass algebra over field K. Then $R = \bigoplus_{i=1}^{r} R_i$, where $R_i = M_{n_i}(D_i)$ for fin dim div algebra D_i,
- R_i are uniquely determined.
- R has exactly r iso classes of simple right modules S_i,
- $\dim_{D_i}(S_i) = n_i$

if K is algebraically closed then $D_i = K$ $\forall i$.

$\mathbb{C}G$ is the direct sum of matrix algebras over $\mathbb{C}$, where the number of matrix algebras is equal to the number of simple modules up to iso.

Corollary 7.14: if G is a finite group, $Z(\mathbb{C}G)$ is an r-dim $\mathbb{C}$-v.s, and

r = # of isomorphism classes of simple modules
= # of conjugacy classes.

8 QUIVERS

Definition 8.1: A quiver Q is a directed (multi)-graph. no restriction on # of arrows between i and j. We also allow loops $\circlearrowleft$

Definition 8.2: A representation M of Q is a direct sum of vector spaces $\oplus M_i$, where i is the label of vertices, together with linear maps $\theta_x : M_i \to M_j$ for each arrow $i \xrightarrow{x} j$

Definition 8.3: A morphism of representations is a collection of linear maps $M_i \to M_i'$ which commute with the linear maps representing the edges.

Definition 8.4: A path of length $\ell \geq 1$ is a concatenation of ℓ compatible arrows.

Definition 8.5: The path algebra kQ is a k-v.s. with basis given by the paths, and the multiplication is given by concatenation of compatible paths. If two paths are incompatible, then their product is zero.

Lemma 8.6:
a) kQ is finite dimensional $\Leftrightarrow$ Q is finite and it contains no directed cycles
b) If Q is finite, then kQ is finitely generated.

Suppose M is a representation of our quiver Q: $M = \oplus M_i$ and if $\exists$ an edge $i \xrightarrow{x} j$, then x acts on M_i by applying θ_x.

Thus $\oplus M_i$ can be thought of as a kQ-module. We get a correspondence
$$\{kQ\text{-modules}\} \longleftrightarrow \{\text{representations of } Q\}$$

$S_i \longleftrightarrow$ representation $:= \begin{cases} k & \text{at vertex } i \\ 0 & \text{otherwise} \end{cases}$ + all maps are zero.

Example: Q finite, no directed cycles, and simple modules as described example. Then these S_i are the only simple modules of kQ.

Definition 8.7: An algebra R has finite representation type if there are only finitely many indecomposable modules (up to isomorphism).

Theorem 8.8 (Gabriel, 1972): let k be an algebraically closed field. A connected quiver has a path algebra of finite representation type if and only if its underlying graph (ignoring directions) is of type A_r $(r \geq 1)$, D_r $(r \geq 4)$, E_6, E_7, E_8 (the simply laced Coexter graphs).

Remarks (1) this is independent of the direction of the arrows.
(2) If we drop the algebraically closed restriction, we can get other coexter graphs, e.g. B_r, C_r, F_4, G_2.
(3) the more general theorem is a classification of <u>pos-def</u> Coexeter graphs.

Given a Coexter graph, we can define a symmetric bilinear form on the $\mathbb{R}$-span of the vertices $v_1, \dots, v_n$ (say), which form a basis for this vector space.

$$q_{ij} = \begin{cases} 2 & \text{if } i = j \\ -\sqrt{t_{ij}} & \text{if } i \neq j \end{cases}$$
where t_{ij} = # of edges connecting the two vertices
no direction in coexeter graph

If a Coexeter graph arises from a root system, say $\Delta = \{\alpha_1, \dots, \alpha_r\}$ of root system Φ, then
$$q_{ij} = \frac{2(\alpha_i, \alpha_j)}{|\alpha_i| \, |\alpha_j|}$$
Symmetrized version of the cartan matrix

Note that this matrix is the same as the one representing the inner product wrt basis $\left\{ \frac{\alpha_i}{|\alpha_i|}, \alpha_i \in \Delta \right\}$. This matrix is therefore positive definite

Definition 8.8: A coexeter graph is positive definite if $[q_{ij}]$ is positive definite.

Lemma 8.9: A connected positive definite Coexeter graph with r vertices has that the number of pairs of vertices joined by at least one edge $= r-1$.

Definition 8.10: The dimension vector of a representation.
$$\sum (\dim M_i) v_i \in \mathbb{R}^r \qquad i = \text{vertices}, \; v_1, \dots, v_r \text{ basis} \subseteq \mathbb{R}^r$$

Theorem 8.12 (Gabriel) Q quiver, underlying Coexeter graph which is simply laced + positive definite. ie. A_r, D_r, E_6, E_7, E_8). Then
$$\left\{ \begin{matrix} \text{Isomorphism classes of finite} \\ \text{dimensional indecomposable} \\ \text{representations of } Q. \end{matrix} \right\} \longleftrightarrow \{\text{positive roots in } \mathbb{R}^r\}$$
dimension vectors $\sum k_i v_i \longleftrightarrow \sum k_i \alpha_i$ $\quad \alpha_i$ simple roots.

Thus kQ has finite representation type.

Lemma 8.16: For a standardised quiver Q
1) $1 \leq j < r$, j is a sink and $j+1$ is a source of $s_j s_{j-1} \cdots s_2 s_1 Q$
2) $1 < j \leq r$, j is a source and $j-1$ is a sink of $s_j s_{j+1} \cdots s_r Q$
3) $s_1 s_2 \cdots s_r Q = s_r s_{r-1} \cdots s_1 Q = Q$

Definition 8.17: numbering of vertices is admissible if for each j, j is a sink of $s_{j+1} s_{j+2} \cdots s_r Q$

Lemma 8.18: There is an admissible numbering for the vertices of Q iff has no directed cycles.

Exercise: Q and Q' with same underlying graph a tree, then $\exists$ $j_1, \dots, j_r$ such that $s_{j_1} \cdots s_{j_s} Q = Q'$.

Definition 8.19 We define functors
$$\mathfrak{S}_j^+ : Q\text{-representations} \longrightarrow s_j Q\text{-representations}$$
$$\mathfrak{S}_j^- : s_j Q\text{-representations} \longrightarrow Q\text{-representations}$$

Given a representation of Q, V, let $\mathfrak{S}_j^+(V) = W$ where $W_i = V_i$ $i \neq j$ and W_j = kernel of $\phi = \oplus$ of maps representing the arrows with target j

The functor $\mathfrak{S}_j^-$ is the dual of this. Given a representation W of $s_j Q$, let $V_i = W_i$ if $i \neq j$. Set V_j = coker of Σ of maps representing arrows with source in $s_j Q$.

8.20 Lemma: s_j^- and s_j^+ give a bijection between
$$\left\{ \begin{matrix} \text{indecomposable rep}^{\text{ns}} \\ \text{of } Q \neq \text{irreducible} \\ \text{rep}^{\text{n}} \text{ of dimension } 1 \\ \text{concentrated at vertex } j \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Indecomposable rep}^{\text{ns}} \\ \text{of } s_j Q \neq \text{irreducible} \\ 1\text{-dim}^{\text{l}} \text{ rep}^{\text{n}} \text{ concentrated} \\ \text{at vertex } j \end{matrix} \right\}$$

8.21 Corollary: kQ has finite rep$^{\text{n}}$ type iff $K s_j Q$ has finite rep$^{\text{n}}$ type

Definition 8.22: A Coxeter element c of the Weyl group $\Phi(\Phi)$ is a product of each simple reflections exactly once, in any order.

Coxeter elts are not unique, but are conjugate $\to$ have same order $h \to$ "cox. number"
$h = \frac{\#\text{ roots}}{\text{rank } r}$, so $\dim$ Lie alg $= \dim(H) + \#\text{roots} = r + hr = (h+1)r$

Definition 8.23:
$1, \dots, r$ an admissible numbering of Q. The Coxeter functor wrt this numbering is
$$C^+ := \mathfrak{S}_1^+ \cdots \mathfrak{S}_r^+ : \underset{\text{of } Q}{\text{reptns}} \longrightarrow \underset{\text{of } Q}{\text{reptns}}$$
$$C^- := \mathfrak{S}_r^- \cdots \mathfrak{S}_1^- : \underset{\text{of } Q}{\text{reptns}} \longrightarrow \underset{\text{of } Q}{\text{reptns}}$$

Lemma 8.24: Given an indecomposable representation V of Q, either
(i) $C^- C^+ (V) = V$, or
(ii) $C^+(V) = 0$

If L is nilpotent and nonzero, then $Z(L) \neq 0$.

By assumption, $L_{(c)} = 0$ for some integer c. Since L is nonzero, $c \neq 1$. So $c \geq 2$

$\Rightarrow$ $[L_{(c-1)}, L] = 0$ for some $c \geq 2$.

$L_{(c-1)}$ is an ideal of L, and pairs with every element in L to zero, so

$L_{(c-1)} \subset Z(L)$.